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Expansion Problems Arising from the Watson Transformation

EUGENE PFLUMM

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Abstract

The Watson transformation permits us to express the radial dependency of certain wave functions in terms of Hankel functions of complex order. These Hankel functions may be considered as "eigenfunctions" of a non-selfadjoint boundary value problem for Bessel's differential operator. It turns out that in general an arbitrary function cannot be expanded in a series involving these eigenfunctions, not even if very strong conditions are imposed on the arbitrary function. However, there exist sufficiently 'weak' non-selfadjoint perturbations of a selfadjoint problem connected with Bessel's operator for which an expansion theorem for an arbitrary function in terms of the eigenfunctions can be proven.

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1. Introduction

In the theory of propagation of electromagnetic waves around the surface of a sphere produced by a unit dipole, the field can be expressed in a series of Hankel functions $H_n^{(1)}(kr)$. These are of integral order n; the variable is kr where k is the wave number and r is the distance from the center of the sphere. If kr is large, this series converges very slowly and is unsuitable for numerical computations. It has been shown by G.N. Watson and others that in this case a convergent expansion for the wave (at least in the "shadow region") can be obtained in the form of an infinite series:

$$u = \sum_{n=1}^{\infty} c_n H_{\nu_n}^{(1)}(kr)$$

where the numbers ν_n are the roots of a transcendental function of ν. In the case of a perfectly conducting sphere, this function is simply

$$H_\nu^{(1)}(ka). \tag{1.1}$$

It is known that there exists an infinitude of complex roots of $H_\nu(ka) = 0$.

It has been pointed out by Sommerfeld [1] that the functions $H_{\nu_n}(kr)$ form an orthogonal system. If we replace kr by a real variable x, and assume that k = 1, we have

$$\int_a^{\infty} H_{\nu_n}^{(1)}(x)\, H_{\nu_m}^{(1)}(x)\, \frac{dx}{x} = 0 \tag{1.2}$$

if $m \neq n$.

Therefore, if a function f(x) can be written in the form

$$f(x) = \sum_{n=1}^{\infty} c_n H_{\nu_n}^{(1)}(x) \tag{1.3}$$

and if the series on the right-hand side converges absolutely and uniformly, we can determine the constants c_n from

$$c_n = \frac{\int_a^\infty f(x) H_{\nu_n}^{(1)}(x) \frac{dx}{x}}{\int_a^\infty \left(H_{\nu_n}^{(1)}\right)^2 \frac{dx}{x}} . \tag{1.4}$$

B. Friedman [2] studied the whole complex of questions arising from the Watson transformation [4]. He showed that the question of whether f(x) can be expanded in a series of the type (1.3) may be considered as a generalized (non-selfadjoint) eigenvalue problem for the second order differential operator L defined by

$$L = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + x^2 \tag{1.5}$$

where the boundary conditions are given for the interval (a,∞). The condition at x = a is an ordinary boundary condition, but the condition at infinity is of the type of a "radiation condition" (see (1.7)). B. Friedman suggested that the problem of expanding f(x) in a series of "eigenfunctions" $H_{\nu_n}^{(1)}(x)$ of this boundary value problem be treated by using the methods employed by Titchmarsh [3]. This method hinges on the proof that a certain integral in the complex ν-plane, taken over a large semi-circle, tends toward zero as the radius of the circle tends toward infinity. It is somewhat surprising that this method does not work in the present case. This fact can be demonstrated indirectly by showing that the integral in the complex ν-plane will not tend toward zero and, directly, by showing that the series (1.3) with coefficients (1.4) will not converge for a "reasonable" function f(x) which satisfies the proper boundary conditions at $x = a$ and $x = \infty$. So far, no set of necessary and sufficient conditions is known which could be used for deciding the question of whether a function f(x) admits an expansion (1.3) (excepting, of course, the trivial condition that f(x) is given as a series of the type (1.3)).

Under these circumstances it seems to be of some interest to find out whether there exist any non-selfadjoint boundary value problems connected with the operator L (and of the type studied by B. Friedman) to which Titchmarsh's method can be applied. This will be done in Section 2, where a dependency of the operator L on the eigenvalue parameter is assumed in such a manner that the problem may be considered as a perturbed Sturm-Liouville Problem. (An ordinary Sturm-Liouville Problem connected with L arises if we consider the interval $(ia, i\infty)$ on the imaginary axis; in this case the eigenvalues lie on the imaginary ν-axis). In the case of the perturbed problem the eigenvalues ν_n^* may have arbitrarily large real parts, but they will not move as far away from the imaginary axis as the zeros ν_n of the function $R_\nu^{(1)}(a)$ for a real value of a.

In Appendix II, we shall show that the series (1.3) with coefficients (1.4) diverges for

$$(1.6) \qquad f_o(x) = \frac{H_\rho^{(1)}(x)}{H_\rho^{(1)}(a)} - \frac{H_\sigma^{(1)}(x)}{H_\sigma^{(1)}(a)}$$

where ρ and σ are different positive numbers. This is true although the function $f_o(x)$ vanishes for $x = a$, is analytic and satisfies the radiation condition

$$(1.7) \qquad \lim_{x \to \infty} \left| \frac{df_o(x)}{dx} - if_o(x) \right| x = 0.$$

2. A 'Perturbed' Sturm-Liouville Problem

Theorem 1.

If $f(x)$ has a continuous second derivative in (a,N) such that $f(x) = 0$ for $x = a$, $a > 0$ and for $x \geqq N$ then in the interval (a,N) $f(x)$ may be written

$$f(x) = \sum c(\nu_n^2)\phi_n(x).$$

$\phi_n(x)$ are the eigenfunctions and the ν_n^2 are the eigenvalues of the differential operator:

$$L = x^2 \frac{d}{dx^2} + x \frac{d}{dx} - x^2[1 + g(\nu)]$$

i.e.

$$L\phi_n(x) = \nu_n^2\phi_n(x).$$

with the boundary conditions

$$\phi_n(a) = 0, \qquad \phi_n(x) \sim \frac{e^{i\alpha_n x}}{\sqrt{x}} \qquad \text{as } x \to \infty.$$

$\alpha = \alpha(\nu)$ is defined by $\alpha = i\sqrt{1 + g(\nu)}$, $\operatorname{Im} \alpha \geqq 0$. $g(\nu)$ is assumed to be an analytic function of ν in $\operatorname{Re} \nu > 0$ such that $\operatorname{Re} g(\nu) + 1 > b > 0$

$$|g(\nu)| \leqq M \quad \text{as } |\nu| \to \infty,$$

$$\left.\begin{array}{l} \operatorname{Im} g(\nu) = \mathcal{O}\left[\dfrac{\log r \log(\log r)}{r}\right] \text{ and } < 0 \\ \\ \dfrac{dg}{d\nu} = \mathcal{O}\left(\dfrac{1}{\nu}\right) \end{array}\right\} \begin{array}{l} \text{in } \frac{\pi}{2} \geq |\Psi| \geq \frac{\pi}{2} - \epsilon \\ \nu = re^{i\Psi}, \ \epsilon > 0 \end{array}$$

$g(i\tau) = g(-i\tau)$ for real τ. See Appendix I.

The eigenfunctions except for a multiplicative function of ν will be

$$H_{\nu_n}^{(1)}(\alpha_n x)$$

i.e. Hankel functions of the first kind with ν_n chosen so that

$$H_{\nu_n}^{(1)}(\alpha a) = 0.$$

We will omit the superscript (1) in what follows and write simply $H_\nu(\alpha x)$.

Proof:

Consider two auxiliary functions $\Psi_1(x)$ and $\Psi_2(x)$ such that

$$(L - \nu^2)\Psi_1(x) = (L - \nu^2)\Psi_2(x) = 0$$

$\Psi_1(a) = 0$ (to satisfy the lower boundary condition)

$\Psi_2(x) = H_\nu(\alpha x)$ (to satisfy the upper boundary condition).

We choose $\Psi_1(x)$ as

$$\Psi_1(x) = J_{-\nu}(\alpha a)J_\nu(\alpha x) - J_\nu(\alpha a)J_{-\nu}(\alpha x).$$

Consider the inhomogeneous equation

$$(L - \nu^2)\,\Phi(x,\nu) = f(x)$$

where $f(x)$ satisfies the conditions of Theorem 1.

$\Phi(x,\nu)$ may be expressed in terms of $\Psi_1(x)$ and $\Psi_2(x)$:

$$\Phi(x,\nu) = \frac{-\pi H_\nu(\alpha x)}{2H_\nu(\alpha a)\sin\pi\nu} \int_a^x \frac{J_{-\nu}(\alpha a)J_\nu(\alpha y) - J_\nu(\alpha a)J_{-\nu}(\alpha y)}{y} f(y)dy$$

$$- \frac{\pi J_{-\nu}(\alpha a)J_\nu(\alpha x) - J_\nu(\alpha a)J_{-\nu}(\alpha x)}{2H_\nu(\alpha a)\sin\pi\nu} \int_x^N \frac{H_\nu(\alpha y)f(y)}{y} dy.$$

Consider

$$- \frac{1}{\pi i} \int_{C_m} \Phi(x,\nu)\nu \, d\nu$$

where C_m is an appropriately chosen sequence of contours in the right half plane. (We may consider them as a sequence of approximate semi-circles, the radii of which tend toward infinity.)

We observe that along the imaginary ν-axis $\nu\Phi(x,\nu)$ is an odd function of ν; hence

$$- \frac{1}{\pi i} \int_{-i\tau}^{i\tau} \Phi(x,\nu)\nu d\nu = 0. \qquad \text{(See App. 1, Sec. III and IV.)}$$

$\Phi(x,\nu)$ is a meromorphic function of ν in the right half-plane. Hence we may evaluate the integrals by considering the residues due to the zeros ν_n of $H_\nu(\alpha a)$. At these points the wronskian of $\Psi_1(x)$ and $\Psi_2(x)$ vanishes. Therefore

$$\Psi_1(x) = k\,\Psi_2(x)$$

where k does not depend on x.

Hence

$$-\frac{1}{\pi i}\int_{C_m}\Phi(x,\nu)\nu d\nu = \sum c_n(\nu_n)H_{\nu_n}(\alpha x) \quad \text{(a finite sum).}$$

On the other hand, we will now show by considering the asymptotic form of $\Phi(x,\nu)$ for large $|\nu|$ that the above integral over the sequence of contours C_m tends toward f(x) as $m \longrightarrow \infty$. For this purpose we will need the following:

Lemma:

For large $|\nu|$ the zeros of $H_\nu(\alpha a)$ in the right half plane have their argument asymptotic to

$$\frac{\pi}{2} - \frac{h(\nu)}{1 + \log\frac{2r}{Aea}}$$

where

$$\alpha = Ae^{i\theta}$$

$$\nu = re^{i\Psi}$$

$$\theta = \frac{\pi}{2} - h(\nu).$$

Proof:

For $|\nu| \longrightarrow \infty$

$$H_\nu(\alpha a) = \frac{J_{-\nu}(\alpha a) - J_\nu(\alpha a)e^{-\pi\nu i}}{i\sin\pi\nu}$$

$$\sim \frac{1}{i\sin\pi\nu}\left[\frac{(\frac{\alpha a}{2})^{-\nu}}{\Gamma(1-\nu)} - \frac{(\frac{\alpha a}{2})^{\nu}e^{-\pi\nu i}}{\Gamma(1+\nu)}\right]$$

and therefore if $H_\nu(\alpha a) = 0$,

$$\left(\frac{\alpha a}{2}\right)^{-2\nu} \nu^{2\nu} e^{-2\nu} \sim \frac{e^{-\pi\nu i}}{2 \sin \pi\nu} .$$

Taking absolute values we see that

$$\left(\frac{Aea}{2r}\right)^{-2r\cos\Psi} e^{2(\theta-\Psi)r\sin\Psi} \sim \left| \frac{1}{e^{2\pi\nu i} - 1} \right| .$$

Since $r^{r\cos\Psi}$ dominates for $r \longrightarrow \infty$ in most of the upper right half plane, it is clear that $\cos \Psi$ must be small, or that $\Psi_0 = \arg \nu$ when $H_\nu(\alpha a) = 0$ is of the form

$$\Psi_0 = \frac{\pi}{2} - \delta$$

where δ is small. Hence we may replace $\cos \Psi_0$ by δ and $\sin \Psi_0$ by 1 and we find

$$\delta \log \frac{2r}{Aea} + \left[\frac{\pi}{2} - h(\nu) - \frac{\pi}{2} + \delta\right] \longrightarrow 0$$

or

$$\Psi_0 \sim \frac{\pi}{2} - \frac{h(\nu)}{1 + \log \left(\frac{2r}{Aea}\right)} . \qquad \text{(cf. App. I, Sec. IV.)}$$

Note that to the right of the ray $\arg \nu = \Psi_0$, $J_{-\nu}(\alpha a)$ is the dominant part of $i \sin \pi\nu \, H_\nu(\alpha a)$, and between the positive imaginary axis and the ray $\Psi = \Psi_0$, $J_\nu(\alpha a)e^{-\pi\nu i}$ is the dominant part.

This leads us to consider a similar ray in the ν-plane on which

$$\left| J_{-\nu}(2N\alpha) \right| = \left| J_\nu(2N\alpha) \, e^{-\pi\nu i} \right| .$$

We have for $|\nu| \longrightarrow \infty$

$$\left|\frac{J_\nu(\alpha x)e^{-\pi\nu i}}{J_{-\nu}(\alpha x)}\right| \sim \left|\left(\frac{\alpha x}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\,e^{-\pi\nu i}\right|$$

$$\sim \left|\pi\left(\frac{\alpha x}{2}\right)^{2\nu}\frac{e^{-\pi\nu i}}{\nu\Gamma^2(\nu)\sin\pi\nu}\right|$$

$$\sim \left|\left(\frac{Aex}{2r}\right)^{2r\cos\Psi}\frac{e^{2r(\Psi-\theta)\sin\Psi}}{e^{2\pi\nu i}-1}\right|$$

$$= \left|\frac{x}{2N}\right|^{2r\cos\Psi}\left(\frac{Ae2N}{2r}\right)^{2r\cos\Psi}\frac{e^{2r(\Psi-\theta)\sin\Psi}}{|e^{2\pi\nu i}-1|}$$

$$\leq \left(\frac{x}{2N}\right)^{2r\cos\Psi} M \qquad \text{(cf. App. I, Sec. I)}$$

where M is independent of both ν and x.

This is valid in the region between the ray described above and the imaginary axis, assuming that ν remains a bounded distance away from the integers. Hence for any $\epsilon > 0$

$$\left|\frac{J_\nu(\alpha x)e^{-\pi\nu i}}{J_{-\nu}(\alpha x)}\right| < \epsilon$$

for all x $\qquad a \leq x \leq N$

on a contour Γ_1 which we will take as a semi-circle of non-integral radius extending from the negative imaginary axis to the line where

$$\left|J_{-\nu}(2N\alpha)\right| = \left|e^{-\pi\nu i}J_\nu(2N\alpha)\right| . \qquad \text{(sufficiently large } \nu\text{)}$$

This simplifies our asymptotic expression for $\Phi(x,\nu)$ since we may replace the Hankel functions by the simpler Bessel functions J_ν on Γ_1. Thus we see that

$$\Phi(x,\nu) \sim \frac{-\pi J_{-\nu}(\alpha x)}{2J_{-\nu}(\alpha a)\sin\pi\nu} \int_a^x \Big(J_{-\nu}(\alpha a)J_\nu(\alpha y) - J_{-\nu}(\alpha y)J_\nu(\alpha a)\Big) \frac{f(y)dy}{y}$$

$$- \frac{\pi\Big(J_{-\nu}(\alpha a)J_\nu(\alpha x) - J_\nu(\alpha a)J_{-\nu}(\alpha x)\Big)}{2J_{-\nu}(\alpha a)\ \sin\pi\nu} \int_x^N \frac{J_{-\nu}(\alpha y)f(y)dy}{y}$$

$$\sim \frac{-1}{2\nu} \int_a^x \left[(\tfrac{y}{x})^{\nu} - (\tfrac{a^2}{xy})\right]^{\nu} \frac{f(y)}{y}\, dy$$

$$- \frac{1}{2\nu} \int_x^N \left[(\tfrac{x}{y})^{\nu} - (\tfrac{a^2}{xy})\right]^{\nu} \frac{f(y)}{y}\, dy\ .$$

Consider

$$\int_a^x (\tfrac{y}{x})^{\nu} \frac{f(y)}{y}\, dy = \frac{1}{\nu}(\tfrac{y}{x})^{\nu} f(y)\Bigg|_{y=a}^{y=x} - \frac{1}{\nu}\int_a^x (\tfrac{y}{x})^{\nu} f'(y)dy.$$

Integrating the last term by parts, we see that

$$\int_a^x (\tfrac{y}{x})^{\nu} \frac{f(y)}{y}\, dy = \frac{1}{\nu} f(x) + \mathcal{O}\left(|\nu|^{-2}\right)\ .$$

We note that the other terms of the asymptotic expansions of the Bessel functions do not contribute a lower order term:

$$J_\nu(\alpha x) = \left(\frac{\alpha x}{2}\right)^\nu \sum_{\ell=0}^{\infty} \frac{\left(\frac{i\alpha x}{2}\right)^{2\ell}}{\ell!\,\Gamma(\nu+\ell+1)} = \frac{\left(\frac{\alpha x}{2}\right)^\nu}{\Gamma(1+\nu)} \left(1 - \frac{\alpha^2 x^2}{(1+\nu)} + \mathcal{O}\left(\frac{1}{|\nu|^2}\right)\right) .$$

Applying a similar argument of integrating by parts to the terms resulting from consideration of these additional terms, we conclude that no $\frac{1}{\nu}$ term, i.e. larger term, enters. Similarly,

$$\int_x^N \left(\frac{x}{y}\right)^\nu \frac{f(y)}{y} = \frac{1}{\nu} f(x) + \mathcal{O}\left(\frac{1}{|\nu|^2}\right)$$

and

$$\int_a^N \left(\frac{x}{y}\right)^\nu \frac{f(y)}{y}\,dy = 0 + \frac{1}{\nu} \int_a^N \left(\frac{a^2}{xy}\right)^\nu f'(y)dy = \mathcal{O}\left(\frac{1}{|\nu|^2}\right) .$$

Hence

$$-\frac{1}{\pi i} \int_{\Gamma_1} \Phi(x,\nu)\nu d\nu \sim \frac{(\pi-\delta')}{\pi} f(x) \longrightarrow f(x)$$

as the radius of the sequence of semi-circles tends towards infinity. δ' is the angle that the ray .

$$\left| J_{-\nu}(2N\alpha) \right| = \left| J_\nu(2N\alpha)e^{-\pi\nu i} \right|$$

makes with the imaginary axis.

We now choose contour Γ_2 so that

$$\arg J_{-\nu}(\alpha a) - \arg\left(J_\nu(\alpha a)e^{-\pi\nu i}\right) = \text{const} \neq 2\pi I$$

where I is any integer.

This will insure the denominator being bounded from below as the contour Γ_2 passes through the region near the zeros of $H_\nu(\alpha a)$. Investigating this further, we find that if

$$\nu = \sigma + i\tau$$

then

$$\arg e^{-\pi\nu i} \left(\frac{\alpha a}{2}\right)^{2\nu} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \sim 2\sigma\theta + 2\tau \log \frac{Aa}{2} - \tau \log(\sigma^2 + \tau^2)$$

$$- 2\sigma \tan^{-1} \frac{\tau}{\sigma} + 2\tau + \frac{\pi}{2}.$$

(We observed that

$$\arg \Gamma(\nu) = \text{Im} \log \Gamma(\nu)$$

and applied Stirling's formula.)

$$\frac{d\tau}{d\sigma} \sim - \frac{- 2 \tan^{-1} \frac{\tau}{\sigma} + 2\theta + 2\sigma \frac{\partial\theta}{\partial\sigma} + \frac{2\tau}{A} \frac{\partial A}{\partial\sigma}}{- \log(\sigma^2 + \tau^2) + \log(\frac{Aa}{2}) + 2\sigma \frac{\partial\theta}{\partial\tau} + \frac{2\tau}{A} \frac{\partial A}{\partial\tau}} \; .$$

This is small. Hence Γ_2 is approximately a straight line, the length of which is

$$\leq \text{const } r\delta' \sim \frac{\text{const } rh(\nu)}{1 + \log \frac{r}{AeN}} \; .$$

Along that part of Γ_2 to the right of the ray where

$$\left|J_{-\nu}(\alpha a)\right| = \left|J_\nu(\alpha a)e^{-\pi\nu i}\right|$$

we can find a constant so that

$$\left|J_{-\nu}(\alpha a) - J_{\nu}(\alpha a)e^{-\pi\nu i}\right| \geqq \text{const}\left|J_{-\nu}(\alpha a)\right|.$$

This we will designate as Γ_3.

The remaining part of Γ_2, denoted by Γ_4, will have the property:

$$\left|J_{-\nu}(\alpha a) - J_{\nu}(\alpha a)e^{-\pi\nu i}\right| \geqq \text{const}\left|J_{\nu}(\alpha a)e^{-\pi\nu i}\right|.$$

Considering as before

$$\left|\frac{J_{\nu}(\alpha x)e^{-\pi\nu i}}{J_{-\nu}(\alpha x)}\right|$$

we conclude on Γ_4:

$$\left|\frac{H_{\nu}(\alpha x)}{H_{\nu}(\alpha a)}\right| \leqq \text{const}\left|\frac{J_{\nu}(\alpha x)}{J_{\nu}(\alpha a)}\right| \leqq \text{const}\left(\frac{x}{a}\right)^{r\delta} .$$

On Γ_3:

$$\left|\frac{H_{\nu}(\alpha x)}{H_{\nu}(\alpha a)}\right| \leqq \max\left(\text{const}\left|\frac{J_{-\nu}(\alpha x)}{J_{-\nu}(\alpha a)}\right| , \text{const}\left|\frac{J_{\nu}(\alpha x)e^{-\pi\nu i}}{J_{-\nu}(\alpha a)}\right|\right)$$

$$\leqq \max\left[\text{const}\left(\frac{x}{a}\right)^{-\text{Re}\,\nu} , \text{const}\left(\frac{x}{a}\right)^{\text{Re}\,\nu}\right]$$

$$\leqq \text{const}\left(\frac{x}{a}\right)^{r\delta'}$$

$$\left(\text{since } \left| \frac{J_\nu(\alpha x)e^{-\pi\nu i}}{J_{-\nu}(\alpha a)} \right| \leq \text{const} \left| \frac{J_\nu(\alpha x)e^{-\pi\nu i}}{J_\nu(\alpha a)e^{-\pi\nu i}} \right| \text{ on } \Gamma_3\right) .$$

Consider

$$\int_a^x \left[(\tfrac{y}{a})^\nu - (\tfrac{a}{y})^\nu\right] \frac{f(y)}{y}\, dy = \frac{1}{\nu}\,(\tfrac{x}{a})^\nu\, f(x) + \frac{1}{\nu}\,(\tfrac{a}{x})^\nu\, f(x) + \mathcal{O}\left(\frac{1}{|\nu|^2}\, M^{\operatorname{Re}\nu}\right)$$

$$= \mathcal{O}\left(\frac{1}{|\nu|}\, M^{|\nu|\delta'}\right)$$

where M is some constant. Hence if $\omega(y,\nu)$ denotes the wronskian of $\Psi_1(y,\nu)$ and $\Psi_2(y,\nu)$, then

$$\left| \int \Gamma_2 \nu d\nu \cdot \Psi_2(x,\nu) \int_a^x \frac{\Psi_1(y,\nu)f(y)}{\omega(y,\nu)y^2}\, dy \right| \leq \text{const } |\nu|\cdot|\nu|\ \delta'\ \frac{1}{|\nu|^2}\, M^{\nu\delta'}$$

$$= \text{const } \delta' M^{\nu\delta'}.$$

This will tend to zero as $|\nu|$ tends toward infinity if

$$\delta' = \mathcal{O}\left(\frac{\log(\log |\nu|)}{|\nu|}\right)$$

since

$$\text{const } \delta' M^{|\nu|\delta'} = \mathcal{O}\left(\frac{\log(\log |\nu|)}{|\nu|}\right)(\log |\nu|)^{\log M}.$$

This will be true when $g(\nu)$ satisfies the conditions of the theorem:

$$\alpha^2(\nu) = -\left(1 + g(\nu)\right) = A^2 e^{2i\theta}.$$

For large $|\nu|$ we have

$$\theta \sim \frac{1}{2} \arctan \frac{\mathrm{Im}\, g(\nu)}{\mathrm{Re}\, g(\nu) + 1} + \frac{\pi}{2}.$$

(We assume that IM $g(\nu)$ is negative in this region.) Since Im $g(\nu)$ is small and Re $g(\nu)$ is bounded away from -1, we have

$$-h(\nu) + \frac{\pi}{2} = \theta \sim \frac{1}{2} \frac{\mathrm{Im}\, g(\nu)}{\mathrm{Re}\, g(\nu) + 1} + \frac{\pi}{2}.$$

Therefore,

$$\delta' \sim \frac{h(\nu)}{1 + \log \frac{r}{AeN}} = \mathcal{O}\left(\frac{\log r \log(\log r)}{r\left(1 + \log \frac{r}{AeN}\right)}\right) = \mathcal{O}\left[\frac{\log(\log r)}{r}\right].$$

We observe that this implies that the real parts of the ν_n may tend toward infinity.

A similar consideration is valid for

$$\int_{\Gamma_2} \nu d\nu \cdot \Psi_1(x,\nu) \int_x^N \frac{\Psi_2(y,\nu) f(y)}{\omega(y,\nu) y^2}\, dy.$$

Therefore we conclude that

$$\lim_{m \to \infty} \frac{-1}{\pi i} \int_{C_m} \Phi(x,\nu)\nu d\nu = f(x).$$

Evaluating the integral by the theorem of residues we have therefore

$$f(x) = \sum C(\nu_n)\, H_{\nu_n}(\alpha x)$$

for any x in the interval (a,N).

(The zeros of sin $\pi\nu$ do not make any contribution to the integral since at the integers

$$J_n(z) = (-1)^n\, J_{-n}(z).$$

Hence at these points the numerator simultaneously vanishes.)

Theorem 2.

If

$$\tilde{L} = L - q(x)x^2$$

where L is the operator of Theorem 1 and q(x) is a continuous function of x in (a,T) such that

$$q(x) = 0 \qquad x \geqq T,$$

then any function f(x) satisfying conditions of Theorem 1 may be expanded in terms of the eigenfunctions of the operator $\tilde{L}$ with the same boundary conditions.

Proof:

We will denote the functions of Theorem 2 that correspond to those of Theorem 1 by a tilde.

Consider $\tilde{\Psi}_1(x,\nu)$ such that

$$(\tilde{L} - \nu^2)\tilde{\Psi}_1(x,\nu) = 0$$

$$\tilde{\Psi}_1(a,\nu) = 0.$$

$\tilde{\Psi}_1(x,\nu)$ may be written in terms of $\Psi_1(x,\nu)$, the auxiliary function used in the proof of Theorem 1:

$$\tilde{\Psi}_1(x,\nu) = \Psi_1(x,\nu) + \frac{\pi}{2\sin\pi\nu}\int_a^x y\Big(J_\nu(\alpha x)J_{-\nu}(\alpha y) - J_{-\nu}(\alpha x)J_\nu(\alpha y)\Big)q(y)\tilde{\Psi}_1(y,\nu)dy.$$

For large $|\nu|$

$$\tilde{\Psi}_1(x,\nu) \sim \frac{(\frac{x}{a})^\nu - (\frac{a}{x})^\nu}{\Gamma(1+\nu)\Gamma(1-\nu)}\cdot\left(1 + \mathcal{O}\left(\frac{1}{|\nu|}\right)\right) + \frac{1}{\nu}\int_a^x y\left[(\tfrac{x}{y})^\nu - (\tfrac{y}{x})^\nu\right]$$

$$\times \left[1 + \mathcal{O}(\tfrac{1}{\nu})\right]q(y)\tilde{\Psi}_1(y,\nu)dy$$

$$\left|(\tfrac{x}{a})^{-\nu}\tilde{\Psi}_1(x,\nu)\right| \leq \frac{4}{|\Gamma(1+\nu)\Gamma(1-\nu)|} + \frac{4}{|\nu|}\int_a^x y\left|(\tfrac{y}{a})^{-\nu}\tilde{\Psi}_1(y,\nu)\right||q(y)|dy.$$

Let

$$M_1 = \max_{a \leqq y \leqq x} \left| \left(\frac{y}{a}\right)^{-\nu} \Psi_1(y,\nu) \right| .$$

Then

$$M_1 \leqq \frac{4}{|\Gamma(1+\nu)\Gamma(1-\nu)|} + \frac{4M_1 Q}{|\nu|}$$

for sufficiently large $|\nu|$, where

$$Q = \int_a^T y|q(y)|dy .$$

Hence

$$M_1 \leqq \frac{\dfrac{4}{|\Gamma(1+\nu)\Gamma(1-\nu)|}}{1 - \dfrac{4Q}{|\nu|}} \leqq \frac{8}{|\Gamma(1+\nu)\Gamma(1-\nu)|} ,$$

assuming $|\nu| \geqq 8Q$.

Substituting this bound, we find that

$$\left|\widetilde{\Psi}_1(x,\nu) - \Psi_1(x,\nu)\right| \leqq \frac{\dfrac{\text{const}}{|\nu|} \left|\left(\frac{x}{a}\right)^{\nu}\right|}{|\Gamma(1+\nu)\Gamma(1-\nu)|} .$$

Hence, except in the vicinity of the imaginary axis, we have

$$\widetilde{\Psi}_1(x,\nu) = \Psi_1(x,\nu) \left(1 + \mathcal{O}\left(\frac{1}{|\nu|}\right)\right) .$$

Similarly, we consider

$$\widetilde{\Psi}_2(x,\nu) = \Psi_2(x,\nu) - \frac{\pi}{2\sin\pi\nu} \int_x^T \left[J_\nu(\alpha x)J_{-\nu}(\alpha y) - J_\nu(\alpha y)J_{-\nu}(\alpha x)\right] yq(y)\widetilde{\Psi}_2(y,\nu)dy .$$

We consider first ν along Γ_1 as defined in the proof of Theorem 1, with N replaced by max (N,T).

$$\tilde{\Psi}_2(x,\nu) \sim \frac{(\frac{\alpha x}{2})^{-\nu}\left(1+o(1)\right)\cdot}{i\Gamma(1-\nu)\ \sin \pi\nu} - \frac{1}{\nu}\int_x^T \left[(\frac{x}{y})^{\nu} - (\frac{y}{x})^{\nu}\right]\left[1 + \sigma(\frac{1}{\nu})\right] yq(y)\tilde{\Psi}_2(y,\nu)dy.$$

$$\left|(\frac{\alpha x}{2})^{\nu}\tilde{\Psi}_2(x,\nu)\right| \leq \left|\frac{2}{\Gamma(1-\nu)\sin\pi\nu}\right| + \frac{4}{|\nu|}\int_x^T \left|(\frac{\alpha y}{2})^{\nu}\tilde{\Psi}_2(y,\nu)\right|\cdot y(q(y)|dy.$$

Let

$$M_2 = \max_{x \leq y \leq T} \left|(\frac{\alpha y}{2})^{\nu}\tilde{\Psi}_2(y,\nu)\right| .$$

Then

$$M_2 \leq \frac{2}{|\Gamma(1-\nu)\ \sin \pi\nu|} + \frac{4QM_2}{|\nu|} .$$

Hence

$$M_2 \leq \frac{\left|\frac{2}{\Gamma(1-\nu)\ \sin \pi\nu}\right|}{1 - \frac{4Q}{|\nu|}} \leq \frac{4}{|\Gamma(1-\nu)\ \sin \pi\nu|}$$

for $|\nu| \geq 8Q$ as before.

Substituting this bound, we obtain:

$$\left|\tilde{\Psi}_2(x,\nu) - \Psi_2(x,\nu)\right| \leq \frac{\text{const}}{|\nu|} \cdot \frac{\left|\left(\frac{\alpha x}{2}\right)^{-\nu}\right|}{|\Gamma(1-\nu)\ \sin \pi\nu|} .$$

Hence along Γ_1, except in the vicinity of the imaginary axis, we have

$$\tilde{\Psi}_2(x,\nu) = \Psi_2(x,\nu)\left(1 + \mathcal{O}\left(\frac{1}{|\nu|}\right)\right).$$

Along Γ_2 we have:

$$\left|\left(\frac{\alpha x}{2}\right)^{\nu}\tilde{\Psi}_2(x,\nu)\right| \leq \frac{2\left|\left(\frac{\alpha x}{2}\right)^{\nu}\right|}{|\sin \pi\nu|} \max_{x \leq y \leq T}\left[\left|\frac{\left(\frac{\alpha y}{2}\right)^{-\nu}}{\Gamma(1-\nu)}\right|, \left|\frac{\left(\frac{\alpha y}{2}\right)^{\nu} e^{-\pi\nu i}}{\Gamma(1+\nu)}\right|\right]$$

$$+ \frac{4}{|\nu|}\int_x^T \left|\left(\frac{\alpha y}{2}\right)^{\nu}\tilde{\Psi}_2(y,\nu)\right| y|q(y)|dy.$$

Hence along Γ_2

$$\left|\tilde{\Psi}_2(x,\nu)\right| \leq \frac{4\left|\left(\frac{T}{x}\right)^{\nu}\right|}{|\sin \pi\nu|} \max_{x \leq y \leq T}\left[\left|\frac{\left(\frac{\alpha y}{2}\right)^{-\nu}}{\Gamma(1-\nu)}\right|, \left|\frac{\left(\frac{\alpha y}{2}\right)^{\nu} e^{-\pi\nu i}}{\Gamma(1+\nu)}\right|\right].$$

We now obtain estimates for the wronskian of $\left(\tilde{\Psi}_1(x,\nu)\tilde{\Psi}_2(x,\nu)\right)$ which will be denoted by $\tilde{\omega}(x,\nu)$.

$$\tilde{\omega}(x,\nu) = \tilde{\Psi}_1(x,\nu)\frac{d\tilde{\Psi}_2(x,\nu)}{dx} - \tilde{\Psi}_2(x,\nu)\frac{d\tilde{\Psi}_1(x,\nu)}{dx}$$

$$= W\left(\Psi_1(x,\nu),\Psi_2(x,\nu)\right)$$

$$+ W\left[\Psi_1(x,\nu) - \frac{\pi}{2\sin \pi\nu}\int_x^T G(x,y)\tilde{\Psi}_2(y,\nu)yq(y)dy\right]$$

$$+ W\left[\frac{\pi}{2\sin\pi\nu}\int_a^x G(x,y)\widetilde{\Psi}_1(y,\nu)yq(y)dy,\ \Psi_2(y,\nu)\right]$$

$$+ W\left[\frac{\pi}{2\sin\pi\nu}\int_a^x G(x,y)\widetilde{\Psi}_1(y,\nu)yq(y)dy,\ -\frac{\pi}{2\sin\pi\nu}\int_x^T G(x,y)\widetilde{\Psi}_2(y,\nu)yq(y)dy\right]$$

$$\equiv W_1 + W_2 + W_3 + W_4,$$

in which $W[\quad]$ denotes the wronskian and

$$G(x,y) = J_\nu(\alpha x)J_{-\nu}(\alpha y) - J_\nu(\alpha y)J_{-\nu}(\alpha x).$$

We observe that

$$W_1 = W(\Psi_1(x,\nu),\Psi_2(x,\nu)) = \omega(x,\nu)$$

is the Wronskian of Theorem 1.

We will now show that

$$\left|\frac{W_2}{W_1}\right|,\quad \left|\frac{W_3}{W_1}\right| \text{ and } \left|\frac{W_4}{W_1}\right|$$

are small along both Γ_1 and Γ_2 and therefore we will be later justified in replacing $\widetilde{\omega}$ by ω.

We observe that

$$W_2 = W\left[\Psi_1(x,\nu),\ \frac{-\pi}{2\sin\pi\nu}\int_x^T G(x,y)\tilde{\Psi}_2(y,\nu)yq(y)dy\right]$$

$$= -\frac{\pi}{2\sin\pi\nu}\int_x^T W\left[\Psi_1(x,\nu),G(x,y)\right]\tilde{\Psi}_2(y,\nu)yq(y)dy$$

$$= +\frac{1}{x}\int_x^T G(a,y)\tilde{\Psi}_2(y,\nu)yq(y)dy.$$

Along Γ_1 for sufficiently large $|\nu|$:

$$\left|\frac{W_2}{W_1}\right| \leq \frac{\text{const}\displaystyle\int_x^T \frac{\left|\left(\frac{y}{a}\right)^{\nu}\right|}{|\Gamma(1+\nu)\Gamma(1-\nu)|}\cdot\frac{\left|\left(\frac{\alpha y}{2}\right)^{-\nu}\right|y|q(y)|dy}{|\Gamma(1-\nu)\ \sin\pi\nu|}}{\dfrac{\left|\left(\frac{\alpha a}{2}\right)^{-\nu}\right|}{|\Gamma(1-\nu)|}}$$

$$\leq \frac{\text{const}}{|\nu|}\ .$$

Along Γ_4 we have:

$$\left|\frac{W_2}{W_1}\right| \leq \frac{\text{const}\displaystyle\int_x^T \frac{\left|\left(\frac{y}{a}\right)^{\nu}\right|}{|\Gamma(1+\nu)\Gamma(1-\nu)|}\cdot\frac{\left|\left(\frac{T}{y}\right)^{\nu}\right|\left|\left(\frac{\alpha T}{2}\right)^{\nu}\right|\ y|q(y)|dy}{|\Gamma(1+\nu)\ \sin\pi\nu|}}{\left|\dfrac{\left(\frac{\alpha a}{2}\right)^{\nu}}{\Gamma(1+\nu)}\right|}$$

$$\leq \frac{\text{const}}{|\nu|}M^{\text{Re}\,\nu}$$ where M is some constant.

We conclude on Γ_4, since $\operatorname{Re}\nu = \mathcal{O}\Big(\log(\log r)\Big)$, that

$$\left|\frac{W_2}{W_1}\right| \longrightarrow 0 \qquad \text{as } |\nu| \longrightarrow \infty .$$

Along Γ_3 we have

$$\left|\frac{W_2}{W_1}\right| \leq \frac{\text{const} \displaystyle\int_x^T \frac{\left|\left(\frac{y}{a}\right)^{\nu}\right| \left|\left(\frac{T}{v}\right)^{\nu}\right|}{|\Gamma(1+\nu)\Gamma(1-\nu)|} \max_{y \leq z \leq T} \left[\left|\frac{\left(\frac{\alpha z}{2}\right)^{-\nu}}{\Gamma(1-\nu)}\right|, \left|\frac{\left(\frac{\alpha z}{2}\right)^{\nu} e^{-\pi\nu i}}{\Gamma(1+\nu)}\right|\right] \frac{y|q(y)|dy}{|\sin \pi\nu|}}{\left|\frac{\left(\frac{\alpha a}{2}\right)^{-\nu}}{\Gamma(1-\nu)}\right|}$$

$$\leq \frac{\text{const}}{|\nu|} M^{\operatorname{Re}\nu} \quad \text{as in the previous case.}$$

Therefore, we conclude that

$$\left|\frac{W_2}{W_1}\right| \longrightarrow 0 \text{ along the entire contour } C_m \text{ as } m \longrightarrow \infty .$$

We treat the ratio

$$\left|\frac{W_3}{W_1}\right| \text{ in a similar manner:}$$

$$W_3 = W\left[\frac{\pi}{2\sin\pi\nu} \int_a^x G(x,y)\tilde{\Psi}_1(y,\nu)yq(y)dy, \Psi_2(x,\nu)\right]$$

$$= \frac{\pi}{2\sin\pi\nu} \int_a^x W\Big[G(x,y), \Psi_2(x,\nu)\Big]\tilde{\Psi}_1(y,\nu)yq(y)dy$$

$$= \frac{1}{x}\int_a^x H_\nu(\alpha y)\widetilde{\Psi}_1(y,\nu)yq(y)dy.$$

We have along Γ_1 for sufficiently large $|\nu|$:

$$\left|\frac{W_3}{W_1}\right| \leq \frac{\text{const}\displaystyle\int_a^x \frac{\left|\left(\frac{\alpha y}{2}\right)^{-\nu}\right|}{|\Gamma(1-\nu)\sin\pi\nu|}\;\frac{\left|\left(\frac{y}{a}\right)^{\nu}|y|q(y)\right|dy}{|\Gamma(1+\nu)\Gamma(1-\nu)|}}{\left|\dfrac{\left(\frac{\alpha a}{2}\right)^{-\nu}}{\Gamma(1-\nu)}\right|}$$

$$\leq \frac{\text{const}}{|\nu|}.$$

Along Γ_3 and Γ_4 we find as in the previous case

$$\left|\frac{W_3}{W_1}\right| \leq \frac{\text{const}}{|\nu|}M^{\mathrm{Re}\,\nu} \qquad \text{for some constant } M.$$

$$W_4 = \frac{-\pi^2}{4\sin^2\pi\nu}W\left[\int_a^x G(x,y)\widetilde{\Psi}_1(y,\nu)yq(y)dy, \int_x^T G(x,y)\widetilde{\Psi}_2(y,\nu)yq(y)dy\right]$$

$$= \frac{\pi}{2x\sin\pi\nu}\left[\left(\int_x^T J_\nu(\alpha y)\widetilde{\Psi}_2(y,\nu)yq(y)dy\right)\cdot\left(\int_a^x J_{-\nu}(\alpha y)\widetilde{\Psi}_1(y,\nu)yq(y)dy\right)\right.$$

$$\left.-\left(\int_a^x J_\nu(\alpha y)\widetilde{\Psi}_1(y,\nu)q(y)ydy\right)\cdot\left(\int_x^T J_{-\nu}(\alpha y)\widetilde{\Psi}_2(y,\nu)q(y)ydy\right)\right].$$

Substituting the bounds as done previously we find:

$$\left|\frac{W_4}{W_1}\right| \leq \frac{\text{const}}{|\nu|^2} \qquad \text{in } \Gamma_1$$

$$\leq \frac{\text{const}}{|\nu|^2} M^{\mathrm{Re}\,\nu} \qquad \text{in } \Gamma_2$$

$$\longrightarrow 0 \qquad \text{along } C_m \text{ as } m \longrightarrow \infty.$$

Therefore, we conclude $\omega(x,\nu) = W_1 + W_2 + W_3 + W_4$ is dominated by $W_1 = \omega(x,\nu)$ and along the entire contour C_m and specifically:

$$\tilde{\omega} = \omega\left(1 + \mathcal{O}\left(\frac{1}{|\nu|}\right)\right) \qquad \text{along } \Gamma_1$$

$$\tilde{\omega} = \omega\left(1 + \mathcal{O}\left(\frac{M^{\mathrm{Re}\,\nu}}{|\nu|}\right)\right) \qquad \text{along } \Gamma_2.$$

Consider the inhomogeneous equation

$$(\tilde{L} - \nu^2)\tilde{\Phi}(x,\nu) = f(x).$$

$\tilde{\Phi}(x,\nu)$ may be expressed in terms of $\tilde{\Psi}_1(x,\nu)$ and $\tilde{\Psi}_2(x,\nu)$:

$$\tilde{\Phi}(x,\nu) = \tilde{\Psi}_2(x,\nu) \int_a^x \frac{\tilde{\Psi}_1(y,\nu)f(y)dy}{\tilde{\omega}(y,\nu)y^2} + \tilde{\Psi}_1(x,\nu) \int_x^N \frac{\tilde{\Psi}_2(y,\nu)f(y)dy}{\tilde{\omega}(y,\nu)y^2} .$$

We will consider this expression for large ν and show that:

$$\lim_{m \longrightarrow \infty} \frac{-1}{\pi i} \int_{C_m} \tilde{\Phi}(x,\nu)\nu d\nu = f(x).$$

Consider

$$\tilde{\Psi}_2(x,\nu) \int_a^x \frac{\tilde{\Psi}_1(y,\nu)f(y)dy}{\tilde{\omega}(y,\nu)y^2}$$

$$= \left[\Psi_2(x,\nu) - \frac{\pi}{2 \sin \pi\nu} \int_x^T G(x,y)\tilde{\Psi}_2(y,\nu)yq(y)dy \right]$$

$$\times \left[\int_a^x \left[\Psi_1(y,\nu) + \frac{\pi}{2 \sin \pi\nu} \int_a^y G(y,z)\tilde{\Psi}_1(z,\nu)zq(z)dz \right] \frac{f(y)dy}{\tilde{\omega}(y,\nu)y^2} \right]$$

$$= \Psi_2(x,\nu) \int_a^x \frac{\Psi_1(y,\nu)f(y)}{\tilde{\omega}(y,\nu)y^2} + \frac{\pi}{2 \sin \pi\nu} \Psi_2(x,\nu) \int_a^x \frac{f(y)dy}{\tilde{\omega}(y,\nu)y^2}$$

$$\times \int_a^y G(y,z)\tilde{\Psi}_1(z,\nu)zq(z)dz$$

$$- \frac{\pi}{2 \sin \pi\nu} \int_x^T G(x,y)\tilde{\Psi}_2(y,\nu)yq(y)dy \cdot \int_a^x \frac{\Psi_1(y,\nu)f(y)dy}{\tilde{\omega}(y,\nu)y^2}$$

$$+ \frac{\pi^2}{4 \sin^2 \pi\nu} \int_x^T G(x,y)\tilde{\Psi}_2(y,\nu)yq(y)dy \cdot \int_a^x \frac{f(y)dy}{\tilde{\omega}(y,\nu)y^2} \int_a^y G(y,z)\tilde{\Psi}_1(z,\nu)zq(z)dz$$

$$\equiv \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \,.$$

We note that

$$\Phi_1 = \Psi_2(x,\nu) \int_a^x \frac{\Psi_1(y,\nu)f(y)}{\tilde{\omega}(y,\nu)y^2}\, dy$$

$$\sim \Psi_2(x,\nu) \int_a^x \frac{\Psi_1(y,\nu)f(y)}{\omega(y,\nu)y^2}\, dy$$

which was considered in proof of Theorem 1. Hence

$$\lim_{m \longrightarrow \infty} \frac{-1}{\pi i} \int_{C_m} \Phi_1 \nu d\nu = \frac{f(x)}{2} .$$

We will now show that the contributions to the contour integral due to Φ_2, Φ_3 and Φ_4 tend toward zero. Consider

$$\Phi_2 = \frac{\pi}{2 \sin \pi\nu} \Psi_2(x,\nu) \int_a^x \frac{f(y)dy}{\tilde{\omega}(y,\nu)y^2} \int_a^y G(y,z)\tilde{\Psi}_1(z,\nu)zq(z)dz.$$

We interchange the order of integration and note that

$y\tilde{\omega}(y,\nu)$

is a function of ν only.

$$\Phi_2 = \frac{\pi}{2 \sin \pi\nu} \frac{\Psi_2(x,\nu)}{y\tilde{\omega}(y,\nu)} \int_a^x \tilde{\Psi}_1(z,\nu)zq(z)dz \int_z^x \frac{G(y,z)f(y)dy}{y} .$$

Consider:

$$\int_z^x \frac{G(y,z)f(y)dy}{y} \sim \frac{1}{\Gamma(1+\nu)\Gamma(1-\nu)} \int_z^x \left[\left(\frac{y}{z}\right)^{\nu} - \left(\frac{z}{y}\right)^{\nu} \right] \frac{f(y)dy}{y}$$

$$\sim \frac{\sin \pi\nu}{\pi\nu^2} \left[\left[\left(\frac{y}{z}\right)^{\nu} + \left(\frac{z}{y}\right)^{\nu} \right] f(y) \Big|_z^x - \int_z^x \left[\left(\frac{y}{z}\right)^{\nu} + \left(\frac{z}{y}\right)^{\nu} \right] f'(y)dy \right]$$

by integrating by parts. Hence

$$\left| \int_z^x \frac{G(y,z)f(y)dy}{y} \right| \leq \frac{\text{const}\ |\sin \pi\nu|}{|\nu|^2} \left| \left(\frac{x}{z}\right)^{\nu} \right| .$$

Therefore along Γ_1

$$|\Phi_2| \leq \frac{\dfrac{\text{const}}{|\sin \pi\nu|} \cdot \dfrac{\left|\left(\frac{\alpha x}{2}\right)^{-\nu}\right|}{|\Gamma(1-\nu)\sin \pi\nu|} \displaystyle\int_a^x \dfrac{\left|\left(\frac{z}{a}\right)^{\nu}\right|}{|\Gamma(1+\nu)\Gamma(1-\nu)|} \cdot \left|\dfrac{\sin \pi\nu}{\nu^2}\right| \cdot \left|\left(\frac{x}{z}\right)^{\nu}\right| dz}{\dfrac{\left|\left(\frac{\alpha a}{2}\right)^{-\nu}\right|}{|\Gamma(1-\nu)|}}$$

$$\leq \frac{\text{const}}{|\nu|^3} .$$

Along Γ_3 we have

$$|\Phi_2| \leq \frac{\dfrac{\text{const}}{|\sin \pi\nu|} \cdot \dfrac{\max}{|\sin \pi\nu|} \left[\dfrac{\left|\left(\frac{\alpha x}{2}\right)^{-\nu}\right|}{|\Gamma(1-\nu)|} , \dfrac{\left|\left(\frac{\alpha x}{2}\right)^{\nu} e^{\pi\nu i}\right|}{|\Gamma(1+\nu)|} \right] \cdot \dfrac{\left|\left(\frac{x}{a}\right)^{\nu}\right|}{|\nu|^3}}{\dfrac{\left|\left(\frac{\alpha a}{2}\right)^{-\nu}\right|}{|\Gamma(1-\nu)|}}$$

$\leq \text{const} \dfrac{M^{\operatorname{Re}\nu}}{|\nu|^3}$ for some constant M.

In like manner we obtain this bound for $|\Phi_2|$ along Γ_4.

Hence

$$\left| \int_{\Gamma_1} \Phi_2 \nu d\nu \right| \leq \frac{\text{const}}{|\nu|}$$

$$\left| \int_{\Gamma_2} \Phi_2 \nu d\nu \right| \leq \text{const} \frac{M^{\mathrm{Re}\,\nu}}{|\nu|}$$

and we conclude that

$$\lim_{m \longrightarrow \infty} \int_{C_m} \Phi_2 \nu d\nu = 0.$$

Similarly we find

$$|\Phi_4| = \left|\frac{\alpha^2 \pi^2}{4 \sin^2 \pi\nu}\right| \left| \int_x^T G(x,y)\widetilde{\Psi}_2(y,\nu) y q(y) dy \right| \cdot \left| \int_a^x \frac{f(y)dy}{\widetilde{\omega}(y,\nu)y^2} \int_a^y G(y,z)\widetilde{\Psi}_1(z,\nu) z q(z) dz \right|$$

$$\leq \frac{\text{const}}{|\nu|^4} \qquad \text{for } \nu \text{ along } \Gamma_1$$

$$\leq \frac{\text{const}}{|\nu|^4} M^{\mathrm{Re}\,\nu} \qquad \text{for } \nu \text{ along } \Gamma_2$$

and

$$|\Phi_3| = \left|\frac{\pi}{2 \sin \pi\nu}\right| \cdot \left| \int_x^T G(x,y)\widetilde{\Psi}_2(y,\nu) y q(y) dy \right| \cdot \left| \int_a^x \frac{\Psi_1(y,\nu) f(y) dy}{\widetilde{\omega}(y,\nu)y^2} \right|$$

$$\leq \frac{\text{const}}{|\nu|^3} \qquad \text{for } \nu \text{ along } \Gamma_1$$

$$\leq \frac{\text{const}}{|\nu|^3} M^{\mathrm{Re}\,\nu} \qquad \text{for } \nu \text{ along } \Gamma_2.$$

We conclude therefore

$$\lim_{m \to \infty} \int_{Cm} (\Phi_3 + \Phi_4)\nu d\nu = 0.$$

In the same manner we find

$$\lim_{m \to \infty} \frac{-1}{\pi i} \int_{Cm} \nu d\nu \,.\, \tilde{\Psi}_1(x,\nu) \int_x^N \frac{\tilde{\Psi}_2(y,\nu)f(y)dy}{\tilde{\omega}(y,\nu)y^2} = \frac{f(x)}{2}$$

and therefore

$$\lim_{m \to \infty} \frac{-1}{\pi i} \int \tilde{\Phi}(x,\nu)\nu d\nu = f(x).$$

As in Theorem 1, we also evaluate this integral by the theorem of residues. Hence we conclude for x in (a,N) that:

$$f(x) = \sum c(\nu_n)\,\tilde{\phi}_n(x).$$

Appendix I

In this appendix modifications of the hypotheses of Theorem 1 will be discussed.

I. Let us consider the following conditions:

1) $\alpha(\nu) = \mathcal{O}(\nu^{1-b})$ for some positive b as $|\nu| \rightarrow \infty$

replacing boundedness.

2) Instead of a wedge we consider the region

$$\frac{\pi}{2} \geq |\Psi| \geq \frac{\pi}{2} - \frac{k}{|\log r|^{1-c}}$$

and

3) $|\alpha'(\nu)| = o\left[\frac{1}{r}(\log r)^{1-c}\right]$ for some positive constants k and c, as $|\nu| \rightarrow \infty$

in this region for the same constant c.

Then we note that the asymptotic formulas for the Bessel functions are still valid for example:

$$J_\nu(\alpha x) = \frac{(\frac{\alpha x}{2})^\nu}{\Gamma(1+\nu)}\left[1 - \frac{\alpha^2 x^2}{4(1+\nu)} + \ldots\right]$$

$$= \frac{(\frac{\alpha x}{2})^\nu}{\Gamma(1+\nu)}\left[1 + \mathcal{O}(|\nu|^{-2b})\right].$$

Furthermore it is necessary to show over Γ_1 that

$$\left|\frac{J_\nu(\alpha x)e^{-\pi\nu i}}{J_{-\nu}(\alpha x)}\right| < \epsilon.$$

$$\left|\frac{J_\nu(\alpha x)e^{-\pi\nu i}}{J_{-\nu}(\alpha x)}\right| = \left(\frac{x}{2N}\right)^{2r\cos\Psi}\left(\frac{AeN}{r}\right)^{2r\cos\Psi} \cdot \frac{e^{2r(\Psi-\theta)\sin\Psi}}{\left|e^{2\pi\nu i} - 1\right|} .$$

This is clearly small in most of the right half plane but near δ', it may not be. In the vicinity of δ' this expression is

$$\sim \left(\frac{x}{2N}\right)^{2r\cos\Psi} \exp\left[2r(-\delta\log\frac{r}{AeN} - \delta + h(\nu))\right].$$

Consider

$$f(\delta) = -\delta\log\frac{r}{AeN} - \delta + h(\nu).$$

We require that f(δ) < 0 for δ > δ' [f(δ') = 0, δ' same as in Theorem 1.]

If

$$\delta \leq \frac{k}{(\log r)^{1-c}} .$$

then clearly f(δ) < 0 for sufficiently large r. For $\delta \leq \frac{k}{(\log r)^{1-c}}$.

We compute the derivative

$$\frac{df(\delta)}{d\delta} = -\log\frac{r}{AeN} + \delta\frac{1}{A}\frac{\partial A}{\partial\delta} - 1 + \frac{\partial h(\nu)}{\partial\delta}$$

and will show that

$$\frac{df(\delta)}{d\delta} < 0$$

in this range.

Consider

$$\alpha(\nu) = A(\nu)\exp\left[i\left(\frac{\pi}{2} - h(\nu)\right)\right]$$

$$\frac{\partial\alpha}{\partial\delta} = - i\nu = \alpha\left(\frac{1}{A}\frac{\partial A}{\partial\delta} - i\,\frac{\partial h(\nu)}{\partial\delta}\right) .$$

Hence

$$\frac{1}{A}\frac{\partial A}{\partial\delta} = - \operatorname{Re}\left[\frac{i\nu\alpha'}{\alpha}\right]$$

$$\frac{\partial h}{\partial\delta} = \operatorname{Im}\left[\frac{i\nu\alpha'}{\alpha}\right] .$$

We conclude that

$$\frac{d}{d\delta}\, f(\delta) < 0$$

when

$$|\alpha'(\nu)| = o\left[\frac{1}{r}(\log r)^{1-c}\right] .$$

We note that this bound for $|\alpha'|$ also suffices to make

$$\frac{d\tau}{d\sigma}$$

small so that we may pass a curve Γ_2 between the zeros and obtain proper estimates.

II. We may relax the condition that $\operatorname{Im}\,\alpha(\nu) \geqq 0$ in the right half plane by merely requiring that

$$\operatorname{Im}\,\alpha(\nu_n) \geqq 0.$$

Furthermore should $\alpha(\nu)$ have an infinite singularity on the imaginary axis then we require the existence of a sequence of curves converging toward the point and on which $\operatorname{Im} \alpha \geqq 0$. To justify this we note

$$\Psi_2(x) = H_\nu(\alpha x) \sim \sqrt{\frac{2}{\pi \alpha x}} \exp\left[i\left(\alpha x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right]$$

$$\Psi_1(x) = J_{-\nu}(\alpha a) J_\nu(\alpha x) - J_\nu(\alpha a) J_{-\nu}(\alpha x) \sim \frac{2}{\pi\alpha} \sqrt{\frac{1}{ax}} \sin \pi\nu \sin \alpha(x-a)$$

for large α.

Consider

$$\Phi(x,\nu) = \Psi_2(x,\nu) \int_a^x \frac{\Psi_1(y,\nu) f(y)}{\omega(y,\nu) y^2} dy + \Psi_1(x,\nu) \int_x^N \frac{\Psi_2(y,\nu) f(y)}{\omega(y,\nu) y^2} dy$$

$$\sim - \frac{e^{i\alpha(x-a)}}{\alpha \sqrt{x}} \int_a^x \frac{\sin \alpha(y-a) f(y)}{y^{3/2}} dy - \frac{\sin \alpha(x-a)}{\alpha \sqrt{x}} \int_x^N \frac{e^{i\alpha(y-a)} + (y)}{y^{3/2}} dy.$$

We observe in the first integral $x \geqq y$ and in the second $y \geqq x$. Hence if we modify the integral along the imaginary axis by replacing sections by the above mentioned sequence of curves on which $\operatorname{Im} \alpha \geqq 0$ we see that the singularity makes no contribution.

III. Should $\alpha(\nu)$ fail to be even on all or part of the Im axis then it is necessary to add to the discrete sum an integral

$$\int_{-\infty}^{\infty} \Phi(x,\tau) \tau d\tau.$$

IV. The condition that

$$\operatorname{Im} g(\nu) < 0$$

near the negative imaginary axis is not necessary. However, should it be positive then additional zeros of $H_\nu(\alpha a)$ will appear in that region. Then the proof would be modified by considering a curve $\tilde{\Gamma}_2$ exactly analagous to Γ_2 but near the negative imaginary axis.

If we replace the $<$ sign by a $\leqq$ sign then zeros of $H_\nu(\alpha a)$ may occur on the imaginary axis. Near such a point we would modify the integral along the imaginary axis by considering a small semicircle in the right half plane centered at that point. This will yield half the 'ordinary' residue.

Appendix II

In this appendix we will show that the Watson series diverges for a rather unexpected case. The function

$$f_o(x) = \frac{H_\rho(x)}{H_\rho(a)} - \frac{H_\sigma(x)}{H_\sigma(a)} ,$$

where ρ and σ are different positive numbers, yields coefficients

$$c(\nu_n) = \frac{\int_a^\infty f_o(x) H_{\nu_n}(x) \frac{dx}{x}}{\int_a^\infty \left(H_{\nu_n}(x)\right)^2 \frac{dx}{x}}$$

such that the series

$$\sum_{n=1}^{\infty} c(\nu_n) H_{\nu_n}(x)$$

diverges.

We note that $f_o(x)$ is an analytic function, and that $f_o(a) = 0$. Also, for large x,

$$H_\rho(x) = \left(\frac{2}{\pi x}\right)^{1/2} \exp\left[i\left(x - \frac{\rho\pi}{2} - \frac{\pi}{4}\right)\right] \left(1 + \sigma\left(\frac{1}{x}\right)\right) .$$

and

$$\frac{dH_\rho(x)}{dx} = i\left(\frac{2}{\pi x}\right)^{1/2} \exp\left[i\left(x - \frac{\rho\pi}{2} - \frac{\pi}{4}\right)\right] \left(1 + \sigma\left(\frac{1}{x}\right)\right) .$$

Hence

$$\left(\frac{dH_\rho}{dx} - i\ H_\rho(x)\right) x = \left(\frac{2}{\pi x}\right)^{1/2} \exp\left[i\left(x - \frac{\rho\pi}{x} - \frac{\pi}{4}\right)\right]\left(i - i + \mathcal{O}\left(\frac{1}{x}\right)\right)\ x$$

$$= \mathcal{O}\left(\frac{1}{x^{1/2}}\right) .$$

Therefore

$$\lim_{x \longrightarrow \infty} \left| \frac{df_o}{dx} - i\ f_o(x) \right| x = 0.$$

In spite of this, we will show that

$$\sum c(\nu_n) H_{\nu_n}(x)$$

diverges.

We can compute the coefficients $c(\nu_n)$ explicitly for $f_o(x)$:

$$\int_a^\infty \frac{H_\rho(x) H_{\nu_n}(x)}{x}\, dx$$

$$= \left. \frac{x\left[H_\rho(x)\frac{d}{dx} H_{\nu_n}(x) - H_{\nu_n}(x)\frac{d}{dx} H_\rho(x)\right]}{\nu_n^2 - \rho^2} \right|_{x = a}^{x \longrightarrow \infty}$$

$$= \frac{-a\ H_\rho(a)\frac{d}{dx} H_{\nu_n}(a)}{\nu_n^2 - \rho^2} ,$$

using the above asymptotic expressions and the fact that $H_{\nu_n}(a) = 0$. Also,

$$\int_a^\infty \frac{\left[H_{\nu_n}(y)\right]^2}{y}\,dy$$

$$= \lim_{\nu \longrightarrow \nu_n} \; - \left. \frac{x\left[H_\nu(x)\frac{d}{dx}H_{\nu_n}(x) - H_{\nu_n}(x)\frac{d}{dx}H_\nu(x)\right]}{\nu_n^2 - \nu^2}\right|_{x=a}$$

$$= \left[\frac{x}{2\nu}\,\frac{d}{dx}H_\nu(x)\cdot\frac{\partial}{\partial\nu}H_\nu(x)\right]_{\substack{x=a\\ \nu=\nu_n}} .$$

Hence

$$\frac{\displaystyle\int_a^\infty \frac{H_\rho(x)H_{\nu_n}(x)}{x}\,dx}{\displaystyle\int_a^\infty \frac{\left[H_{\nu_n}(x)\right]^2}{x}\,dx} = -\frac{2\nu_n}{\nu_n^2 - \rho^2}\;\frac{H_\rho(a)}{\left[\frac{\partial}{\partial\nu}H_\nu(a)\right]_{\nu=\nu_n}} .$$

Therefore

$$c(\nu_n) = \frac{\displaystyle\int_a^\infty \frac{f_o(x)H_{\nu_n}(x)}{x}\,dx}{\displaystyle\int_a^\infty \frac{\left[H_{\nu_n}(x)\right]^2}{x}\,dx} = \frac{2\nu_n}{\left.\frac{\partial}{\partial\nu}H_\nu(a)\right|_{\nu=\nu_n}}\left(\frac{\sigma^2 - \rho^2}{(\nu_n^2 - \rho^2)(\nu_n^2 - \sigma^2)}\right) .$$

In order to estimate the $c(\nu_n)$ and $H_{\nu_n}(x)$, we need the followin[illegible]

Lemma:

The zeros ν_n of $H_\nu(a)$ lie on a ray

$$\arg \nu_n \sim \frac{\pi}{2}\left(1 - \frac{1}{1 + \log \frac{2|\nu_n|}{ae}}\right) .$$

Proof:

For large ν

$$i \sin \pi\nu\, H_\nu(a) \sim \frac{(\frac{a}{2})^{-\nu}}{\Gamma(1-\nu)} - \frac{(\frac{a}{2})^{\nu} e^{-\pi\nu i}}{\Gamma(1+\nu)} .$$

Equating and applying Stirling's formula, we find that

$$(\frac{ae}{2})^{-2\nu_n} \nu_n^{2\nu_n} \sim \frac{e^{-\pi\nu_n i}}{2 \sin \pi\, \nu_n} ,$$

and

$$(\frac{ae}{2})^{-2r\cos\Psi} r^{2r\cos\Psi} e^{-2r\Psi\sin\Psi} \sim \left| \frac{e^{-\pi\nu i}}{2 \sin \pi\nu} \right| ,$$

in which we set

$$\nu_n = re^{i\Psi}.$$

This can be satisfied when $\cos \Psi$ is small. Furthermore, we note Ψ must be close to $+ \pi/2$. We set

$$\delta = \frac{\pi}{2} - \Psi$$

and find that

$$\exp\left\{2r\left[\delta \log \frac{2r}{ae} - \left(\frac{\pi}{2} - \delta\right)\right]\right\} \sim 1$$

or

$$\delta \sim \frac{\frac{\pi}{2}}{1 + \log \frac{2r}{ae}}$$

or

$$\arg \nu_n \sim \frac{\pi}{2}\left(1 - \frac{1}{1 + \log \frac{2r}{ae}}\right) .$$

We will now estimate $H_{\nu_n}(x)$:

$$H_{\nu_n}(x) = \frac{1}{i \sin \pi\nu_n}\left[\frac{J_{-\nu_n}(x)}{\Gamma(1-\nu_n)} - \frac{J_{\nu_n}(x)e^{-\pi\nu_n i}}{\Gamma(1+\nu_n)}\right]$$

$$\sim \frac{1}{i \sin \pi\nu_n}\left[\frac{(\frac{x}{a})^{-\nu_n}(\frac{a}{2})^{-\nu_n}}{\Gamma(1-\nu_n)} - \frac{(\frac{x}{a})^{\nu_n}(\frac{a}{2})^{\nu_n}e^{-\pi\nu_n i}}{\Gamma(1+\nu_n)}\right]$$

$$\sim \frac{1}{i \sin \pi\nu_n} \cdot \frac{(\frac{a}{2})^{\nu_n}e^{-\pi\nu_n i}}{\Gamma(1+\nu_n)}\left[(\frac{x}{a})^{-\nu_n} - (\frac{x}{a})^{\nu_n}\right]$$

(since $H_{\nu_n}(a) = 0$).

$$\frac{(\frac{a}{2})^{\nu_n}}{\Gamma(1+\nu_n)} \sim \frac{1}{\sqrt{2\pi}}(\frac{a}{2})^{\nu_n}\nu_n^{-\nu_n-1/2}e^{\nu_n}$$

$$\sim \frac{1}{\sqrt{2\pi}}\nu_n^{-1/2}(\frac{ae}{2\nu_n})^{\nu_n} \sim \text{const } \nu_n^{-1/2}.$$

Therefore:

$$H_{\nu_n}(x) \sim \text{const}\ \nu_n^{-1/2} \left(\frac{x}{a}\right)^{\nu_n}.$$

We now obtain estimates for

$$\frac{\partial}{\partial\nu} H_\nu(a)\Big|_{\nu = \nu_n}$$

$$H_\nu(a) \sim - i\sqrt{\frac{2}{\pi\nu}}\left[\left(\frac{xe}{2\nu}\right)^{-\nu} + i\left(\frac{xe}{2\nu}\right)^{\nu}\right]$$

and

$$\frac{\partial H_\nu(a)}{\partial\nu}\Bigg|_{\nu=\nu_n} \sim - i\sqrt{\frac{2}{\pi\nu_n}}\left[\left(\frac{xe}{2\nu_n}\right)^{-\nu_n} - i\left(\frac{xe}{2\nu_n}\right)^{\nu_n}\right]\cdot\left[1 - \log\left(\frac{xe}{2\nu_n}\right)\right]$$

$$\sim \text{const}\ \nu_n^{-1/2} \log \nu_n.$$

Therefore

$$c(\nu_n)H_{\nu_n}(x) = \frac{2\nu_n H_{\nu_n}(x)}{\frac{\partial}{\partial\nu} H_\nu(a)\Big|_{\nu=\nu_n}}\left[\frac{\sigma^2 - \rho^2}{(\nu_n^2 - \rho^2)(\nu_n^2 - \sigma^2)}\right]$$

$$\sim \text{const}\ \frac{1}{\nu_n^4 \log \nu_n}\left(\frac{x}{a}\right)^{\nu_n},$$

which is unbounded for all $x > a$. We conclude that the series

$$\sum c_n H_{\nu_n}(x)$$

must diverge for all $x > a$.

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